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Inverse Coefficient Bounds for Bounded Turning Functions in the Cardioid Domain

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Abstract

The primary aim of this article is to obtain the constraints for specific inverse initial coefficients for a subclass of bounded turning functions, which are associated with the cardioid domain and are denoted by $\mathcal{BT}_{car}$. The study also focuses on Fekete-Szegö inequality and the Hankel determinants of order two and three within a specific subclass of functions.

Keywords: coefficient bounds; inverse function; univalent function; Hankel determinant.

1 Introduction and Definitions

Let $\mathcal{H}(\mathbb{D})$ represent the set of holomorphic (analytic) functions within $\mathbb{D}$, the unit disk and the unit disk $\mathbb{D}$ consists of all points $z \in \mathbb{C}$ such that $|z| < 1$. Let,

$$\mathcal{A} := \Big\{ f(z) \in \mathcal{H}(\mathbb{D}) : f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \; z \in \mathbb{D}, f(0) = f'(0) - 1 = 0 \Big\}, \tag{1}$$

and

$$\mathcal{S} = \Big\{ f(z) \in \mathcal{A} : f(z_1) = f(z_2) \implies z_1 = z_2 \Big\}. \tag{2}$$

Consider an analytic function u defined in $\mathbb{D}$ such that $u(0) = 0$ and $|u(z)| < 1$ for all $z \in \mathbb{D}$. If we have,

$$F_1(z) = F_2(u(z)) \quad \text{for} \quad z \in \mathbb{D},$$

then, F_1 is subordinate to F_2, denoted as $F_1 \prec F_2$. Here, u is known as a Schwarz function.

In 1985, De Branges [9] made a significant breakthrough by proving the Bieberbach conjecture. He showed that for $f \in \mathcal{S}$, the coefficients a_n satisfy the inequality $|a_n| \leq n, \forall\, n \geq 2$. This finding provided a strong mathematical foundation to understand the coefficients of univalent functions in complex analysis.

The researchers examined a wide range of subclasses of $\mathcal{S}$ associated with various picture domains before Bieberbach could solve the conjecture. The families of starlike and convex functions, represented by $\mathcal{S}^*$ and $\mathcal{C}$ respectively, are among the most familiar subsets of the set of all univalent functions $\mathcal{S}$. The subclasses $\mathcal{S}^*$ and $\mathcal{C}$ are defined as follows:

$$\mathcal{S}^* = \Big\{ f \in \mathcal{S} : \Re\Big(\frac{zf'(z)}{f(z)}\Big) > 0, \quad z \in \mathbb{D} \Big\},$$

$$\mathcal{C} = \Big\{ f \in \mathcal{S} : \Re\Big(\frac{(zf'(z))'}{f'(z)}\Big) > 0, \quad z \in \mathbb{D} \Big\}.$$

By defining $f(z) = z$, we establish the subclass $\mathcal{BT}$ of bounded turning functions as follows:

$$\mathcal{BT} = \{ f \in \mathcal{S} : f'(z) > 0 \text{ for all } z \in \mathbb{D} \}.$$

In [24], using subordination, Ma and Minda classes $\mathcal{S}^*(\varphi)$, $\mathcal{C}(\varphi)$, and $\mathcal{BT}(\varphi)$ are defined in the following manner:

$$\mathcal{S}^*(\varphi) = \Big\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \phi(z) \Big\},$$

$$\mathcal{C}(\varphi) = \Big\{ f \in \mathcal{A} : \frac{(zf'(z))'}{f'(z)} \prec \phi(z) \Big\},$$

$$\mathcal{BT}(\varphi) = \Big\{ f \in \mathcal{A} : f'(z) \prec \varphi(z) \Big\}.$$

In recent years, researchers have explored other image domains $\varphi(D)$ and studied several classes of univalent functions, including $\mathcal{C}(\varphi)$, $\mathcal{S}^*(\varphi)$, and $\mathcal{BT}(\varphi)$. For instance, the class $\mathcal{S}^*_L = \mathcal{S}^*(\sqrt{1+z})$ defined by fixing $\varphi(z) = (1+z)^{1/2}$ was investigated in [34] by Sokol and Stankiewicz. The class $\mathcal{S}^*_{e^z}$ has been introduced and studied in [25]. Cho et al. [8] studied a class $\mathcal{S}^*_{\sin}$ by selecting $\varphi(z) = 1 + \sin z$. Sakar and Güney [30] studied the m-fold symmetric analytic functions and determined coefficients for analytic bi-univalent functions using fractional calculus through the application of Faber polynomial expansion. In 2021, Barukab et al. [5] studied the Hankel determinant of order three for,

$$\mathcal{R}_s := \left\{f \in \mathcal{A} : zf'(z) \prec 1 + \sinh^{-1} z, \forall\, z \in \mathbb{D}\right\}.$$

Sharma et al. [31] introduced a subclass of starlike functions $\mathcal{S}^*_{car}$ defined by,

$$\mathcal{S}^*_{car} := \left\{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2,\ z \in \mathbb{D}\right\}.$$

The class of starlike functions characterized by,

$$\mathcal{S}^* := \left\{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec 1 + ze^z = \wp(z), \quad \forall\, z \in \mathbb{D}\right\},$$

was introduced and examined by Kumar et al. [18]. Srivastava et al. [33] recently investigated precise constraints for the coefficients of a subclass of functions with bounded turning within a cardioid-shaped domain.

The Hankel determinant $\Psi_{q,m}(f)$, for $q, m \in \mathbb{N}$, involving the coefficients of the function $f \in \mathcal{S}$, is given by,

$$\Psi_{q,m}(f) = \begin{vmatrix} a_m & a_{m+1} & \cdots & a_{m+q-1} \\ a_{m+1} & a_{m+2} & \cdots & a_{m+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m+q-1} & a_{m+q} & \cdots & a_{m+2q-2} \end{vmatrix}.$$

This determinant was studied by Pommerenke [26, 27]. We can obtain different determinants by modifying the parameters q and m, as shown below,

$$\begin{aligned} \Psi_{2,1}(f) &= a_3 - a_2^2, \\ \Psi_{2,2}(f) &= a_2a_4 - a_3^2, \\ \Psi_{3,1}(f) &= 2a_2a_3a_4 - a_3^3 - a_4^2 + a_3a_5 - a_2^2a_5. \end{aligned} \tag{3}$$

These determinants are commonly designated as the first, second, and third determinants of the Hankel matrix. It plays an important role in the study of power series with integral coefficients and singularities [10] . Research has been done on the upper bound of $|\Psi_{q,m}(f)|$ for different subclasses of univalent functions. The Hankel determinant $\Psi_{2,1}(f)$ is recognized as the Fekete-Szegö inequality. The subclass of normalized analytic functions whose positive derivative has a positive real part was studied in [14] . The constraints of the functional $\Psi_{2,2}(f)$ obtained in [15] for each of the sets $\mathcal{C}$ and $\mathcal{S}^*$ were sharp. Rehman et al. [28] estimated the second order Hankel determinant for the subclass of bi-close-to- convex function of complex order.

The estimation of $\Psi_{3,1}(f)$ seems to be little harder as compared to $\Psi_{2,2}(f)$. In 2010, Babalola [4] established the maximum value of $|\Psi_{3,1}(f)|$ for the families $\mathcal{C}$, $\mathcal{S}^*$ and $\mathcal{BT}$. The upper bound for the third Hankel determinant of Basilevic functions was derived in [2]. In the subsequent years, numerous researchers were able to obtain non-sharp constraints for $|\Psi_{3,1}(f)|$ of various subclasses of univalent functions. The sharp estimates of determinant over the class $\mathcal{S}^*$ was investigated by Cho et al. [7]. A subclass of starlike function connected with a domain bounded by an epicycloid with $n-1$ cusps was determined in [32].

The work [22] also investigated sharp extremum for the Hankel determinant of order three, for specific subclasses of convex functions, starlike functions, and functions for bounded turning.

The sharp bound of the Hankel determinant of the third kind for starlike functions of order $\frac{1}{2}$ was investigated [21]. Zaprawa et al. [36] also studied the third Hankel determinant for starlike functions. Hankel determinants for starlike and convex functions associated with sigmoid functions was studied [29]. In [35], the third and fourth Hankel determinants were derived for the class of functions with bounded turning related to Bernoulli's lemniscate in the year 2022.

The $1/4$-theorem of Koebe guarantees the existence of the inverse function, f^{-1} for every univalent function f defined in the unit disc $\mathbb{D}$, and its Taylor series representation is,

$$f^{-1}(w) := w + \sum_{n=2}^{\infty} B_n w^n, \quad (|w| < 1/4). \tag{4}$$

We get from the equation $f(f^{-1}(w)) = w$ that,

$$\begin{aligned} B_2 &= -a_2, \\ B_3 &= -a_3 + 2a_2^2, \\ B_4 &= -a_4 + 5a_2a_3 - 5a_2^3, \\ B_5 &= -a_5 + 6a_2a_4 - 21a_2^2a_3 + 3a_3^2 + 14a_2^4. \end{aligned} \tag{5}$$

Studying the behaviour of the inverse function has drawn more attention from scholars in recent years. Libera et al. [23] established a connection between the coefficients of f and f^{-1} for functions f as in (1), assuming $f(\mathbb{D})$ forms a convex region. Conversely, Kapoor and Mishra [16] expanded upon the findings of Krzyz et al. [17] by establishing upper limits for the initial coefficients of the inverse function f^{-1} when $f \in \mathcal{S}^*(\alpha)$ with $0 \leq \alpha < 1$. Ali [1] obtained accurate limits for the initial coefficients of inverse functions for the class of extremely starlike functions. Gandhi [13] defined another type of starlike functions with,

$$\mathcal{S}^*_{3l} := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec 1 + \frac{4}{5}z + \frac{1}{5}z^4,\ z \in \mathbb{D} \right\}.$$

In a similar manner, Arif et al. [3] examined a subclass of bounded turning functions characterized as,

$$\mathcal{BT}_{3l} := \left\{ f \in \mathcal{A} : f'(z) \prec 1 + \frac{4}{5}z + \frac{1}{5}z^4,\ z \in \mathbb{D} \right\}.$$

In their work, Kumar et al. [19] provided an estimation for the optimal upper bound of the third Hankel determinant for the inverse of functions whose derivatives have a positive real part.

Inspired by Srivastava et al. [33], we now analyse a subclass of bounded turning functions defined by:

$$\mathcal{BT}_{car} := \{ f \in \mathcal{A} : f'(z) \prec 1 + ze^z := \wp(z),\ z \in \mathbb{D} \},$$

in order to determine the sharp bounds on initial coefficients for the inverse, the Fekete-Szegö inequality, and $|\Psi_{3,1}(f^{-1})|$ for functions in the class $\mathcal{BT}_{car}$. Geometrically, each $f \in \mathcal{BT}_{car}$ maps the open unit disk into a cardoid-shaped domain, which is symmetric about the real axis, as shown in Figure 1.

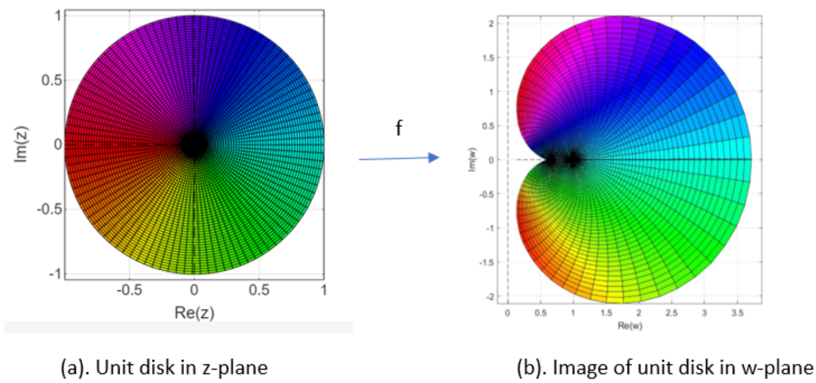


Figure 1: Cardoid domain $\wp(z) = 1 + ze^{z}$.

Obviously the class $\mathcal{BT}_{car}$ is not empty as $f(z) = z \in \mathcal{A}$ will be a trivial member of $\mathcal{BT}_{car}$. That is because by the definition of subordination we can find a function $\phi(z)$ such that $f'(z) = \wp(\phi(z)) = 1 + \phi(z)e^{\phi(z)}$ with $\phi(z) = 0$. As an example, consider the function $f(z) = z + az^2 \in \mathcal{A}$. Then, $f'(z) = 1 + 2az$. Figure 2 is the pictorial representation of $f'(z) \prec \wp(z)$ with $a = \dfrac{1}{6}$.

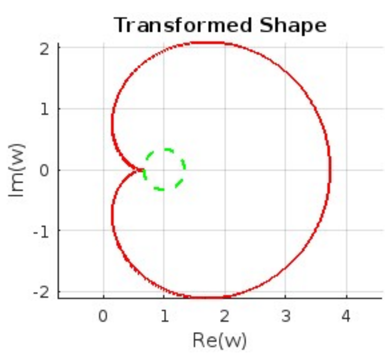


Figure 2: Pictorial representation of $f'(z) \prec \wp(z)$ when $a = 1/6$.

The function $\wp(z) = 1 + ze^{z}$ is holomorphic as it is a combination of holomorphic functions. The derivative of $\wp(z)$ is given by,

$$\wp'(z) = (1 + z)e^{z}.$$

In the unit disk $\mathbb{D}$, neither $e^{z} = 0$ nor $z = -1$ and so $\wp'(z)$ is never zero in $\mathbb{D}$. By the inverse function theorem, this implies that $\wp(z)$ is injective in $\mathbb{D}$. Since $\wp(z) = 1 + ze^{z}$ is both holomorphic and injective in the unit disk $\mathbb{D}$, it follows that $\wp(z)$ is univalent in $\mathbb{D}$.

Also choosing $z = e^{i\theta}, \theta \in [0, 2\pi]$,

$$\Re(\wp'(z)) = \Re(e^{\cos\theta + i\sin\theta}(1 + \cos\theta + i\sin\theta)),$$
$$\Re(\wp'(z)) = \Re(e^{\cos\theta}e^{i\sin\theta}(1 + \cos\theta + i\sin\theta)) > 0.$$

If $f \in BT_{car}$ and if $\Re(\wp(z)) > 0$ in $\mathbb{D}$, then $\Re(f'(z) > 0$ in $\mathbb{D}$ [11] (i.e.) $f(z)$ belongs to a subclass of the close-to-convex functions. Therefore, $f(z)$ is univalent in $\mathbb{D}$ and consequently $f^{-1}(z)$ exists for all $z \in \mathbb{D}$.

2 Preliminaries

Let,

$$P = \Big\{p(z) = 1 + \sum_{n=2}^{\infty} c_n z^n \in \mathcal{H}(\mathbb{D}), \;\; z \in \mathbb{D}, \;\; \Re(p(z)) > 0\Big\}. \tag{6}$$

Consider $p \in \mathcal{P}$ as a Carathéodory type function. The fundamental lemmas outlined below form the foundation of our findings.

Lemma 2.1. *Carathodory's Lemma[6]. Given that $p \in \mathcal{P}$ has the expansion in (6),*

$$|c_t| \leq 2, \tag{7}$$

for $t \geq 1$.

Lemma 2.2. [20] *Let $p \in \mathcal{P}$ be obtained from (6) with $c_1 \geq 0$. Then, we have,*

$$2c_2 = c_1^2 + b(4 - c_1^2), \tag{8}$$

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 b - c_1(4 - c_1^2)b^2 + 2(4 - c_1^2)(1 - |b|^2)\delta, \tag{9}$$

$$8c_4 = c_1^4 + (4 - c_1^2)b\Big[c_1^2\Big(b^2 - 3b + 3\Big) + 4b\Big] - 4(4 - c_1^2)(1 - |b|^2)\Big[c_1(b-1)\delta + \bar{b}\delta^2 - (1 - |\delta|^2)\rho\Big], \tag{10}$$

with $|b| \leq 1, |\delta| \leq 1$ and $|\rho| \leq 1$.

Following is the lemma stated as Lemma 2 in [12].

Lemma 2.3. [12] *Let $\omega(z) = \sum_{n=1}^{\infty} w_n z^n$ be a Schwarz function. Using real numbers β and v, we obtain,*

$$\Pi(\omega) = \big|w_3 + \beta w_1 w_2 + v w_1^3\big| \leq \Pi(\beta, v). \tag{11}$$

where $\Pi(\beta, v)$ is expressed as,

$$\Pi(\beta, v) = \begin{cases} 1, & (\beta, v) \in \mathbb{D}_1 \bigcup \mathbb{D}_2 \bigcup \{(2,1)\}, \\ |v|, & (\beta, v) \in \bigcup_{k=3}^{7} \mathbb{D}_k, \\ \dfrac{2}{3}(|\beta| + 1)\sqrt{\dfrac{|\beta| + 1}{3(|\beta| + 1 + v)}}, & (\beta, v) \in \mathbb{D}_8 \bigcup \mathbb{D}_9, \\ \dfrac{1}{3}v(\dfrac{\beta^2 - 4}{\beta^2 - 4v})\sqrt{\frac{\beta\beta^2 - 4}{3(v-1)}}, & (\beta, v) \in \mathbb{D}_{10} \bigcup \mathbb{D}_{11} \setminus \{(2,1)\}, \\ \dfrac{2}{3}(|v| - 1)\sqrt{\dfrac{\beta - 1}{3(|\beta| - 1 - v)}}, & (\beta, v) \in \mathbb{D}_{12}, \end{cases} \tag{12}$$

and

$$\mathbb{D}_1 = \{(\beta,\nu) : |\beta| \le \frac{1}{2}, \ -1 \le \nu \le 1\},$$
$$\mathbb{D}_2 = \left\{(\beta,\nu) : \frac{1}{2} \le |\beta| \le 2, \ \frac{4}{27}(|\beta|+1)^3 - (|\mu|+1) \le \nu \le 1\right\},$$
$$\mathbb{D}_3 = \{(\beta,\nu) : |\beta| \le \frac{1}{2}, \ \nu \le -1\},$$
$$\mathbb{D}_4 = \{(\beta,\nu) : |\beta| \ge \frac{1}{2}, \ \nu \le -\frac{2}{3}(|\beta|+1)\},$$
$$\mathbb{D}_5 = \{(\beta,\nu) : |\beta| \le 2, \ \nu \ge 1\},$$
$$\mathbb{D}_6 = \left\{(\beta,\nu) : 2 \le |\beta| \le 4, \ \nu \ge \frac{1}{12}(\beta^2+8)\right\},$$
$$\mathbb{D}_7 = \left\{(\beta,\nu) : |\beta| \ge 4, \ \nu \ge \frac{2}{3}(|\beta|-1)\right\},$$
$$\mathbb{D}_8 = \left\{(\beta,\nu) : \frac{1}{2} \le |\beta| \le 2, \ -\frac{2}{3}(|\beta|+1) \le \nu \le \frac{4}{27}(|\beta|+1)^3 - (|\beta|+1)\right\},$$
$$\mathbb{D}_9 = \left\{(\beta,\nu) : |\beta| \ge 2, \ -\frac{2}{3}(|\beta|+1) \le \nu \le \frac{2|\beta|(|\beta|+1)}{\beta^2+2|\beta|+4}\right\},$$
$$\mathbb{D}_{10} = \left\{(\beta,\nu) : 2 \le |\beta| \le 4, \ \frac{2|\beta|(|\beta|+1)}{\beta^2+2|\beta|+4} \le \nu \le \frac{1}{12}\beta^2+8\right\},$$
$$\mathbb{D}_{11} = \left\{(\beta,\nu) : |\beta| \ge 4, \ \frac{2|\beta|(|\beta|+1)}{\beta^2+2|\beta|+4} \le \nu \le \frac{2|\beta|(|\beta|-1)}{\beta^2-2|\beta|+4}\right\},$$
$$\mathbb{D}_{12} = \left\{(\beta,\nu) : |\beta| \ge 4, \ \frac{2|\beta|(|\beta|-1)}{\beta^2-2|\beta|+4} \le \nu \le \frac{2}{3}(|\beta|-1)\right\}.$$

Lemma 2.4. [12] *For every $p \in \mathcal{P}$ and given any complex number κ, it follows that,*

$$\left|c_{i+j} - \kappa c_i c_j\right| \le 2\max\{1, |2\kappa-1|\}. \tag{13}$$

3 Bounds of Inverse Coefficients for the Family $\mathcal{BT}_{car}$

The sharp bounds of the inverse initial coefficients for the functions in the class $\mathcal{BT}_{car}$ are discussed in this section.

Theorem 3.1. *If the function $f \in \mathcal{BT}_{car}$ can be expressed as in (1).Then,*

$$\left|B_2\right| \le \frac{1}{2}, \quad \left|B_3\right| \le \frac{1}{3}. \tag{14}$$

These bounds are sharp.

Proof. By the definition of $f \in \mathcal{BT}_{car}$ and by applying the subordination principle, there is a Schwarz function that fulfills the condition,

$$f'(z) \prec 1 + \omega(z)e^{\omega(z)}, \quad z \in \mathbb{D}.$$

Following this assumption,

$$\omega(z) = \omega_1 z + \omega_2 z^2 + \omega_2 z^3 + \ldots, \qquad z \in \mathbb{D}, \tag{15}$$

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots, \qquad z \in \mathbb{D}. \tag{16}$$

Clearly, we have $p \in \mathcal{P}$ and,

$$\omega(z) = \frac{p(z)-1}{p(z)+1} = \frac{c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \ldots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \ldots}, \quad z \in \mathbb{D} \tag{17}$$

Using (1), it is noted that,

$$f'(z) = 1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + 5a_5 z^4 + \ldots \tag{18}$$

By simple computation and using (17), we get,

$$1 + \omega(z)e^{\omega(z)} = 1 + \frac{1}{2}c_1 z + \frac{1}{2}c_2 z^2 + \left(-\frac{1}{16}c_1^3 + \frac{1}{2}c_3\right) z^3 + \left(\frac{1}{24}c_1^4 - \frac{3}{16}c_1^2 c_2 + \frac{1}{2}c^4\right) z^4 + \ldots \tag{19}$$

By comparing (18) and (19), we get,

$$a_2 = \frac{1}{4}c_1, \tag{20}$$

$$a_3 = \frac{1}{6}c_2, \tag{21}$$

$$a_4 = \frac{1}{8}\left(-\frac{1}{8}c_1^3 + c_3\right), \tag{22}$$

$$a_5 = \frac{1}{10}\left(\frac{1}{12}c_1^4 - \frac{3}{8}c_1^2 c_2 + c_4\right). \tag{23}$$

Substituting (20)–(23) in (5), we get,

$$B_2 = -\frac{1}{4}c_1. \tag{24}$$

$$B_3 = \frac{1}{8}c_1^2 - \frac{1}{6}c_2. \tag{25}$$

$$B_4 = \frac{5}{4}c_1 c_2 - \frac{4}{64}c_1^3 - \frac{1}{8}c_3. \tag{26}$$

$$B_5 = \frac{11}{480}c_1^4 - \frac{29}{160}c_1^2 c_2 + \frac{3}{16}c_1 c_3 + \frac{1}{12}c_2^2 - \frac{1}{10}c_4. \tag{27}$$

The bounds for B_2 and B_3 directly follow from Lemma 2 [12]. Bounds of B_2 and B_3 are illustrated in Figure 3. □

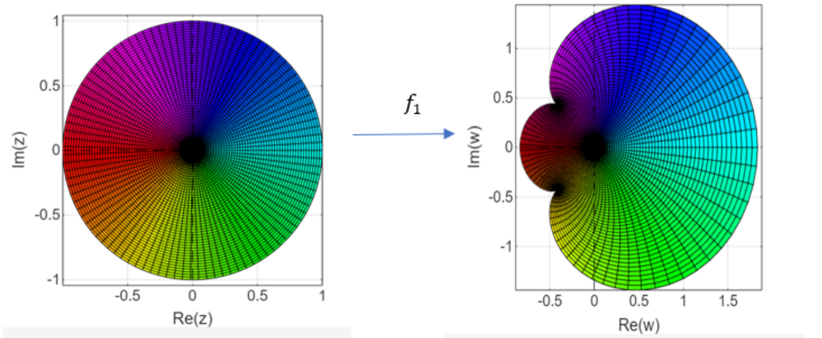
Theorem 3.2. *Let f represented by (1) belong to $\mathcal{BT}_{car}$, then, for $\gamma \in \mathbb{C}$,*

$$\left|B_3 - \gamma B_2^2\right| \leq \frac{1}{3} \max\left\{1, \left|\frac{2-3\gamma}{4}\right|\right\}. \tag{28}$$

This inequality is sharp given by,

$$f_1(z) = \int_0^z (1 + te^t)dt = z + \frac{1}{2}z^2 + \frac{1}{3}z^3 + \ldots., \quad z \in \mathbb{D}, \tag{29}$$

$$f_2(z) = \int_0^z (1 + t^2 e^{t^2})dt = z + \frac{1}{3}z^3 + \ldots., \quad z \in \mathbb{D}. \tag{30}$$



(a). Unit disk in z-plane (b). Image of unit disk in w-plane

Figure 3: Pictorial representation of $f_1(z) = z + 0.333z^2 + 0.2105z^3 + \dots$.

Proof. Employing (24) and (25), we may write,

$$\left|B_3 - \gamma B_2^2\right| = \frac{1}{6}\left|\left(c_2 - 3\left(\frac{2-\gamma}{8}\right)c_1^2\right)\right|. \tag{31}$$

Usage of (13) leads to,

$$\left|B_3 - \gamma B_2^2\right| \leq \frac{1}{3}\max\left\{1, \left|\frac{2-3\gamma}{4}\right|\right\},$$

and the conclusion follows directly. □

Applying $\gamma = 1$, the inequality is derived as mentioned below.

Corollary 3.1. *If the series expansion of f is (1) and f is in $\mathcal{BT}_{car}$, then,*

$$\left|B_3 - B_2^2\right| \leq \frac{1}{3}. \tag{32}$$

Theorem 3.3. *Let f represented by (1) belong to $\mathcal{BT}_{car}$, then, the following inequality is satisfied,*

$$\left|B_2B_3 - B_4\right| \leq \frac{5}{8}.$$

Proof. Using (24) and (25), we have,

$$\left|B_2B_3 - B_4\right| = \frac{1}{8}\left|c_3 - \frac{4}{3}c_1c_2 + \frac{1}{4}c_1^3\right|. \tag{33}$$

From (19) and (20) it is noted that,

$$c_1 = 2\omega_1, \tag{34}$$
$$c_2 = 2(\omega_2 + \omega_1^2), \tag{35}$$
$$c_3 = 2(\omega_3 + 2\omega_1\omega_2 + \omega_1^3). \tag{36}$$

Hence, we obtain,

$$\left|B_2B_3 - B_4\right| = \frac{2}{8}\left|\omega_3 - \frac{2}{3}\omega_1\omega_2 + 5\omega_1^3\right|.$$

Taking $\beta = \dfrac{-2}{3}; \upsilon = \dfrac{5}{2}$, it is understood that $(\beta, \upsilon) \in \mathbb{D}_5$. Using Lemma 2.4, we easily obtain,

$$\left|B_2B_3 - B_4\right| \leq \frac{5}{8}.$$

□

Theorem 3.4. *Let f represented by (1) belong to $\mathcal{BT}_{car}$, then,*

$$\left|\Psi_{2,2}(f^{-1})\right| \leq \frac{1}{9}.$$

The sharpness is given by (29).

Proof. It is to be noted that,

$$\Psi_{2,2}(f^{-1}) = \left|B_2B_4 - B_3^2\right|. \tag{37}$$

From (24)−(26) we have,

$$\Psi_{2,2}(f^{-1}) = -\frac{1}{96}c_1^2c_2 + \frac{1}{32}c_1c_3 - \frac{1}{36}c_2^2. \tag{38}$$

Utilizing (8) and (9) to represent c_2 and c_3 in terms of c_1, and noting that c_1 can be denoted as c where $0 \leq c \leq 2$, the expression obtained is,

$$\left|\Psi_{2,2}(f^{-1})\right| = \frac{1}{4}\left|\frac{-5c^4}{288} - \frac{4c^2x(4-c^2)}{288} - \frac{c^2(4-c^2)x^2}{32} + \frac{2c(4-c^2)(1-|x^2)\sigma}{32} - \frac{x^2(4-c^2)^2}{36}\right|. \tag{39}$$

Next, considering $|\sigma| \leq 1, |x| = t \leq 1$ and taking $c \in [0, 2]$, we apply the triangle inequality to obtain,

$$\begin{aligned}\left|\Psi_{2,2}(f^{-1})\right| &= \frac{1}{4}\left\{\frac{5c^4}{288} + \frac{4c^2t(4-c^2)}{288} + \frac{c^2(4-c^2)t^2}{32} + \frac{2c(4-c^2)(1-|t^2)\sigma}{32} + \frac{t^2(4-c^2)^2}{36}\right\} \\ &= \Phi(c, t).\end{aligned} \tag{40}$$

When differentiating with respect to the parameter t, we get,

$$\frac{\partial\Phi}{\partial t} = \frac{1}{4}\left\{\frac{c^2(4-c^2)}{72} + \frac{tc^2(4-c^2)}{144} - \frac{tc(4-c^2)}{8} + \frac{t(4-c^2)^2}{18}\right\} \geq 0, \tag{41}$$

where $t \in [0, 1]$, it follows that $\Phi(c, t) \leq \Phi(c, 1)$. Substituting $t = 1$ yields,

$$\left|\Psi_{2,2}(f^{-1})\right| \leq \frac{1}{4}\left\{\frac{5c^4}{288} + \frac{13c^2(4-c^2)}{288} + \frac{(4-c^2)^2}{36}\right\} := \chi(c). \tag{42}$$

Clearly $\chi'(c) \leq 0$. This indicates that χ is a function that decreases over $[0, 2]$. Its maximum value is reached for $c = 0$. This implies,

$$\left|\Psi_{2,2}(f^{-1})\right| \leq \frac{1}{9}. \tag{43}$$

□

Theorem 3.5. *If $f \in \mathcal{BT}_{car}$ has the form* (1), *then,*

$$|\Psi_{3,1}(f^{-1})| \leq 0.03807.$$

Proof. According to the definition, $\Psi_{3,1}(f^{-1})$ can be expressed as,

$$\Psi_{3,1}(f^{-1}) = 2B_2B_3B_4 - B_3^3 - B_4^2 + B_3B_5 - B_2^2B_5.$$

In virtue of (24)−(27), we obtain,

$$\Psi_{3,1}(f^{-1}) = \frac{1}{69120}\Big[- 36c_1^6 + 33c_1^4c_2 + 270c_1^3c_3 - 432c_4c_1^2 + 720c_1c_2c_3 - 640c_2^3 - 1080c_3^2 + 1152c_4c_2 - 72c_1^2c_2^2\Big].$$

Utilizing (8)–(10), and performing basic algebraic calculations, we obtain,

$$\begin{aligned}\Psi_{3,1}(f^{-1}) = \frac{1}{69120}\Bigg\{ & \frac{329}{8}c_1^6 + 288t^2x^3 - 80t^3x^3 + 72c_1^2tx^2 + 18c_1^4tx^3 - \frac{981}{4}c_1^4tx^2 + 366c_1^4tx \\ & + \frac{441}{8}c_1^2t^2x^4 - \frac{477}{2}c_1^2t^2x^3 + \frac{141}{2}c_1^2t^2x^2 - \frac{135}{2}t^2\delta^2(1-|x|^2) \\ & + 72c_1^2t(1-|x|^2)(1-|\delta|^2)\rho - 72c_1^3tx(1-|x|^2)\delta - 72c_1^2t\bar{x}(1-|x|^2)\delta^2 \\ & + \frac{639}{2}c_1^3t(1-|x|^2)\delta + 333c_1t^2x^2(1-|x|^2)\delta - 288t^2|x|^2(1-|x|^2)\delta^2 \\ & - \frac{887}{4}c_1t^2x(1-|x|^2)\delta + 288t^2x(1-|x|^2)(1-|\delta|^2)\rho\Bigg\}.\end{aligned}$$

From the above expression, we have,

$$\begin{aligned}\Psi_{3,1}(f^{-1}) = \frac{1}{69120}\Bigg\{ & \frac{329}{8}c_1^6 + t\Big[366c_1^4x - \frac{981}{4}c_1^4x^2 + 72c_1^2x^2 + 18c_1^4x^3 + \frac{141}{2}c_1^2tx^2 \\ & + t\Big[\frac{-447}{2}c_1^2x^3 + \frac{441}{8}c_1^2x^4 + 288x^3 - 80tx^3] \\ & + \Big[\Big(\frac{639}{2} - 72x\Big)c_1^3 - c_1tx\Big(\frac{887}{4} - 333x\Big)\Big](1-|x|^2))\delta \\ & + \frac{441}{2}\Big[\frac{-16}{49}c_1^2\bar{x} - t\Big(|x|^2 + \frac{15}{49}\Big)\Big](1-|x|^2))\delta^2 \\ & + 72\Big[4tx + c_1^2\Big](1-|x|^2)(1-|\delta|^2)\rho\Bigg\}.\end{aligned}$$

Denoting $t = 4 - c^2, c_1 \leftrightarrow c$ and $|\rho| \leq 1$, it is noted that,

$$\begin{aligned}\Psi_{3,1}(f^{-1}) = \frac{1}{69120}\Bigg[& \frac{329}{8}c^6 + (4-c^2)\Bigg\{366c^4x - \frac{981}{4}c^4x^2 + 18c^4x^3 + 72c^2x^2 \\ & + (4-c^2)\Big[- 32x^3 + \frac{441}{8}c^2x^4 - \frac{287}{2}x^3c^2 + \frac{141}{2}c^2x^2\Big]\end{aligned}$$

$$+\left[\left(\frac{639}{2}-72x\right)c^3-c(4-c^2)x\left(\frac{887}{4}-333x\right)\right](1-|x|^2)\delta$$
$$+\frac{441}{2}\left[\frac{-16}{49}c_1^2\bar{x}-(4-c^2)(|x|^2+\frac{15}{49})\right](1-|x|^2)\delta^2+72\left[4tx+c_1^2\right](1-|x|^2)(1-|\delta|^2)\rho\Bigg\}, \tag{44}$$

in which $x, \delta, \rho \in \bar{\mathbb{D}}$, and taking modulus on both sides in above expression, we obtain,

$$\left|\Psi_{3,1}(f^{-1})\right| \leq \frac{\mu(c,x,\delta)}{69120}, \tag{45}$$

where

$$\mu(c,x,\delta)=\frac{329}{8}c^6+(4-c^2)\Bigg\{366c^4x+c^2\left(72-\frac{981}{4}c^4\right)x^2+18c^4x^3$$
$$+(4-c^2)\left[\frac{441}{8}c^2x^4+\left(32+\frac{287}{2}c^2\right)x^3+\frac{141}{2}c^2x^2\right]$$
$$+\left[\left(\frac{639}{2}-72x\right)c^3+c(4-c^2)x\left(\frac{887}{4}-333x\right)\right](1-x^2)\delta$$
$$+\frac{441}{2}\left[\frac{16}{49}c^2x+(4-c^2)\left(x^2+\frac{15}{49}\right)\right](1-x^2)\delta^2+72\left[c^2+4x(4-c^2)\right](1-x^2)(1-\delta^2)\Bigg\}. \tag{46}$$

Now, we maximize the function $\Phi(c,x,\delta)$ within the parallelepiped region given by $[0,2]\times[0,1]\times[0,1]$, where x is in the interval $[0,1]$ and $\delta\in[0,1]$.

A) On the vertices of the parallelepiped, we get,

$$\mu(0,0,0)=\mu(2,0,0)=\mu(2,1,0)=\mu(0,0,2)=2632,$$
$$\mu(0,0,1)=1080,\ \mu(0,1,0)=\mu(0,1,1)=\mu(0,0,2)=512.$$

B) Next, taking into account the eight edges of the parallelepiped,

(i) When $x=1$; $\delta=0$; and $\delta=1$ in (46),

$$\mu(c,1,\delta)=512+4466c^2-1670c^4+\frac{343}{2}\leq 2632.$$

(ii) When $c=0$ and $\delta=0$, we have the following,

$$\mu(0,x,0)=1152x-640x^3\leq 2632, \qquad \text{for } x\in(0,1).$$

(iii) For $x=0$ and $\delta=1$ we get,

$$\mu(c,1,\delta)=\frac{329}{8}+(4-c^2)\left[\frac{639}{2}c^3+\frac{441}{2}(4-c^2)\frac{15}{49}\right]$$
$$=\frac{329}{8}c^6+\frac{135}{2}c^4+1278c^3-540c^2+1080-\frac{639}{2}c^5\leq 2632.$$

(iv) When $c = 0$ and $\delta = 1$, we have,

$$\mu(0, x, 1) = 270 + 612x^2 + 512x^3 - 882x^4 \leq 2632.$$

(v) For $c = 0$ and $x = 0$, we obtain,

$$\mu(0, 0, \delta) = 1080\delta^2 \leq 1080, \text{ for } \delta \in (0, 1)$$

(vi) Putting $c = 0$ & $x = 1$, we obtain,

$$\mu(0, 1, \delta) = 512.$$

(vii) When $c = 2$; for $\delta = 0$ & $\delta = 1$; $x = 0$; & $x = 1$ we have,

$$\mu(c, x, \delta) = 2632.$$

(viii) $x = 0$ and $\delta = 0$.

$$\begin{aligned}\mu(c, 0, 0) &= \frac{329}{8}c^6 + 288c^2 - 72c^4, \ for\ c \in (0, 2)\\ &\leq 2632.\end{aligned}$$

C) We will now analyze the six faces of the parallelepiped.

(i) When $c = 2$, then $\mu(2, x, \delta) = 2632$ for $x, \delta \in (0, 1)$.

(ii) $c = 0$ in (46) leads to,

$$\mu(0, x, \delta) = 512x^3 + \Big[882x^2 + 270 + 1152x\Big](1 - x^2)(1 - \delta^2) \leq 2632.$$

(iii) $x = 0$ in (46), then,

$$\begin{aligned}\mu(c, 0, \delta) &= \frac{329}{8}c^6 + (4 - c^2)\Big[\frac{639}{2}c^3\delta + \frac{441}{2}(4 - c^2)\frac{15}{49}\delta^2 + 72(1 - \delta^2)\Big]\\ \mu(c, 0, \delta) &= \frac{329}{8}c^6 + (4 - c^2)\left[\frac{639}{2}c^3\delta + 270\delta^2 - \frac{135}{2}c^2\delta^2 + 72c^2 - 72c^2\delta^2\right]\\ &= \frac{329}{8}c^6 + (4 - c^2)\left[\frac{639}{2}c^3\delta + 270\delta^2 + c^2\left(72 - \frac{279}{2}\delta^2\right)\right]\\ &\leq \frac{329}{8}c^6 + (4 - c^2)\left[\frac{639}{2}c^3 + 270 + 72c^2\right] \leq 2632.\end{aligned}$$

(iv) Fixing $x = 1$ in (46), with c in $(0, 2)$ and $\delta \in (0, 1)$, it is evident that the function $\mu(c, 1, \delta)$ is not dependent on δ, as observed in $B(i)$.

$$\mu(c, 1, \delta) \leq 2632.$$

(v) If we take $\delta = 0$, with $c \in (0,2)$ and $\delta \in (0,1)$ in (46),

$$\begin{aligned}
\mu(c,x,0) =& \frac{329}{8}c^6 + (4-c^2)\Bigg\{366c^4x + c^2\left(72 - \frac{981}{4}c^2\right)x^2 + 18c^4x^3 \\
&+ (4-c^2)\left[\frac{441}{2}c^2x^4 + \left(32 - \frac{287}{2}c^2\right)x^3 + \frac{141}{2}c^2x^2\right] \\
&+ 72\left[c^2 + 4x(4-c^2)\right](1-x^2)\Bigg\} \\
=& \frac{329}{8}c^6 + (4-c^2)\Bigg\{1152x - 1152x^3 + c^4\left[366x - \frac{441}{2}x^4 - \frac{1263}{4}x^3 - \frac{105}{2}x^2\right] \\
&+ c^2\left[282x^2 + 882x^4 - 318x^3 - 288x + 72\right]\Bigg\} \\
\leq& \frac{329}{8}c^6 + (4-c^2)\left\{\frac{-891}{4}c^4 + 636c^2\right\} \\
\leq& 2632.
\end{aligned}$$

(vi) On the face $\delta = 1$ and for $x \in (0,1)$, from (46), we have,

$$\begin{aligned}
\mu(c,x,1) =& \frac{329}{8}c^6 + (4-c^2)\Bigg\{366c^4x + c^2\left(72 - \frac{981}{4}c^2\right)x^2 + 18c^4x^3 \\
&+ (4-c^2)\left[\frac{441}{8}c^2x^4 + \left(32 - \frac{287}{2}c^2\right)x^3 + \frac{141}{2}c^2x^2\right] \\
&+ \left[\left(\frac{639}{2} - 72c^2\right)c^3 + c(4-c^2)x\left(\frac{887}{4} - 333x\right)\right](1-x^2) \\
&+ \frac{441}{2}\left[\frac{16}{49}c^2x + (4-c^2)\left(x^2 + \frac{15}{49}\right)\right](1-x^2)\Bigg\} \\
=& g(c,x), \text{ with } c \in (0,2) \text{ and } x \in (0,1).
\end{aligned}$$

A simple numerical computation gives the solution of the system $\frac{\partial g}{\partial c} = 0$ and $\frac{\partial g}{\partial x} = 0$ that lies within the region $(0,2) \times (0,1)$. Consequently, there are no critical points when $\delta = 1$.

D) Examining the interior of the parallelepiped with dimensions $(0,2) \times (0,1) \times (0,1)$. We have $\frac{\partial \phi}{\partial \delta} = 0$ if and only if,

$$\delta_0(c,x) = \frac{\frac{639}{2}c^3 - 72c^2x + 1332cx^2 - 887cx - 333c^3x^2 + \frac{887}{4}c^3x}{40c^2x - c^2 - 32x^3c^2 + 127x^3 + 98x^2 - 128x - \frac{11}{2}},$$

for $(c,x) \in (0,2) \times (0,1)$ and $40c^2x - c^2 - 32x^3c^2 + 127x^3 + 98x^2 - 128x - \frac{11}{2} \neq 0$. Thus, $\mu(c,x,1)$ exhibits no critical points within the interior of the parallelepiped. Upon reviewing above four cases, we find that,

$$\max\{\mu(c,x,1) : (c,x) \in [0,2],\ x \in [0,1],\ \delta \in [0,1]\} = 2632.$$

Expressions (45) and (46) yield the following,

$$\left|\Psi_{3,1}(f^{-1})\right| \leq 0.03807.$$

□

4 Conclusion

Although extensive research exists on Hankel determinants in geometric function theory, determining the precise bound for the third Hankel determinant remains challenging. In this paper, we examine a family of bounded turning functions, denoted by $\mathcal{BT}_{car}$, which are associated with the cardoid domain. We addressed the challenge by obtaining exact results for the coefficients of the inverses of these functions. This finding enhances our comprehension of the geometric features of this type of functions. By refining current methodologies, we expect similar results on various known subclasses of univalent functions. We would like to point out that the figures presented in this work were generated using MATLAB.

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